Chapter 1

Quantitative estimates

In this chapter we state our main results concerning with quantitative estimates of the convergence in Trotter's approximation theorem.

These estimates are related to the convergence of sequences of iterates of linear operators to an assigned semigroup, and to the resolvent operator of semigroups's generator.

We make no restriction on the general assumptions in the classical Trotter's Theorem and we only require a quantitative estimates of the voronovskaja-type formula in terms of suitable seminorms.

The results in this chapter are collected in [36, 37, 38].

1.1 Quantitative approximation of semigroups

We need a slight improvement of [64, Lemma III.5.1, p. 89]. On one hand this will provide us better constants in the subsequent main result and on the other hand it will be essential in order to avoid restrictions on the domain of the resolvent operator in the next section.

Lemma 1.1.1 *Let $L : E \to E$ be a linear operator on a Banach space E and assume that there exist $M > 0$ and $N \geq 1$ such that, for every $k \geq 1$,*

$$\|L^k\| \leq MN^k .$$

Then for every $u \in E$ and $k \geq 1$ we have

$$\|e^{k(L-I)}u - L^k u\| \leq M \left(N^{k-1}\sqrt{\frac{2k}{\pi}} + \frac{e^{k(N-1)} - N^k}{N-1} \right) \|Lu - u\| \qquad (1.1.1)$$

if $N > 1$ and

$$\|e^{k(L-I)}u - L^k u\| \leq M \sqrt{\frac{2k}{\pi}} \|Lu - u\| \qquad (1.1.2)$$

if $N = 1$.

Proof. Let $N > 1$; if $0 \le i < k$, we have

$$\begin{aligned}
\|L^k u - L^i u\| &= \sum_{j=i}^{k-1} \|L^{j+1}u - L^j u\| = \sum_{j=i}^{k-1} \|L^j(Lu-u)\| \\
&\le M\,\|Lu-u\| \sum_{j=i}^{k-1} N^j \\
&= M\,\|Lu-u\| \left(\frac{1-N^k}{1-N} - \frac{1-N^i}{1-N} \right) \\
&= M\,\frac{N^k - N^i}{N-1}\,\|Lu-u\|
\end{aligned}$$

and further

$$\frac{N^k - N^i}{N-1} = \sum_{j=i}^{k-1} N^j \le (k-i)N^{k-1} .$$

Similarly, if $0 \le k < i$,

$$\|L^k u - L^i u\| \le M\,\frac{N^i - N^k}{N-1}\,\|Lu-u\| .$$

Therefore we have

$$\begin{aligned}
&\|e^{k(L-I)}u - L^k u\| = \|e^{-k} \sum_{i=0}^{\infty} \frac{k^i}{i!}(L^i u - L^k u)\| \\
&\le e^{-k} \sum_{i=0}^{\infty} \frac{k^i}{i!}\,\|L^i u - L^k u\| \\
&\le e^{-k} \left(\sum_{i=0}^{k-1} \frac{k^i}{i!}\,\|L^i u - L^k u\| + \sum_{i=k+1}^{\infty} \frac{k^i}{i!}\,\|L^i u - L^k u\| \right) \\
&\le M\,\frac{e^{-k}}{N-1} \left(\sum_{i=0}^{k-1} \frac{k^i}{i!}\,(N^k - N^i) + \sum_{i=k+1}^{\infty} \frac{k^i}{i!}\,(N^i - N^k) \right) \|Lu-u\| \\
&= M\,\frac{e^{-k}}{N-1} \left(2\sum_{i=0}^{k-1} \frac{k^i}{i!}\,(N^k - N^i) + \sum_{i=0}^{\infty} \frac{k^i}{i!}\,(N^i - N^k) \right) \|Lu-u\| \\
&= M\,e^{-k} \left(2\sum_{i=0}^{k-1} \frac{k^i}{i!}\,\frac{N^k - N^i}{N-1} + \frac{1}{N-1}(e^{Nk} - N^k e^k) \right) \|Lu-u\| \\
&\le M\,e^{-k} \left(2\sum_{i=0}^{k-1} \frac{k^i}{i!}(k-i)N^{k-1} + \frac{e^{Nk} - N^k e^k}{N-1} \right) \|Lu-u\| \\
&= M\,e^{-k} \left(2N^{k-1} \left(k\sum_{i=0}^{k-1} \frac{k^i}{i!} - \sum_{i=1}^{k-1} \frac{k^i}{(i-1)!} \right) + \frac{e^{Nk} - N^k e^k}{N-1} \right) \\
&\quad \times \|Lu-u\|
\end{aligned}$$

$$= M\,e^{-k}\left(2N^{k-1}\left(k\sum_{i=0}^{k-1}\frac{k^i}{i!} - k\sum_{i=1}^{k-1}\frac{k^{i-1}}{(i-1)!}\right) + \frac{e^{Nk}-N^k e^k}{N-1}\right)$$
$$\times \|Lu-u\|$$
$$= M\,e^{-k}\left(2N^{k-1}k\frac{k^k}{k!} + \frac{e^{Nk}-N^k e^k}{N-1}\right)\|Lu-u\| \,.$$

Applying Stirling formula $k! = \sqrt{2\pi k}\,k^k\,e^{-k}\,e^{\theta_k/(12k)}$, $0 \le \theta_k \le 1$, we obtain

$$\|e^{k(L-I)}u - L^k u\|$$
$$\le M\,e^{-k}\left(2N^{k-1}k\frac{k^k}{\sqrt{2\pi k}\,k^k e^{-k}} + \frac{e^{Nk}-N^k e^k}{N-1}\right)\|Lu-u\|$$
$$= M\left(N^{k-1}\sqrt{\frac{2k}{\pi}} + \frac{e^{k(N-1)}-N^k}{N-1}\right)\|Lu-u\|$$

and this yields (1.1.1). Finally (1.1.2) can be similarly shown using the inequality $\|L^k u - L^i u\| \le M(k-i)\|Lu-u\|$ whenever $0 \le i < k$. □

Observe that (1.1.2) can be obtained from (1.1.1) taking the limit as $N \to 1^+$.

Now we can state one of the main results. Starting with a sequence of linear operators and the generator of a C_0-semigroup satisfying a quantitative Voronovskaja-type formula, we can evaluate the norm difference between the $k(n)$-iterate of the n-th linear operator and the semigroup. Suitable choices of the sequence $(k(n))_{n\in\mathbb{N}}$ ensure the convergence of the iterates to the semigroup.

Namely, let $(L_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on a Banach space E and $A : \mathcal{D} \to E$ a linear operator satisfying the hypotheses of Trotter's Theorem, i.e. the stability condition

$$\|L_n^k\| \le M e^{\omega\, k/n}\,, \tag{1.1.3}$$

and the Voronovskaja-type formula

$$Af = \lim_{n\to\infty} n(L_n f - f)\,.$$

Moreover, assume that $\mathcal{D}$ is a dense subspace of E and $(\lambda - A)(\mathcal{D})$ is dense in E for some $\lambda > \omega$. Then from the classical Trotter's theorem (see [70, Theorem 5.1] or Theorem II.1.1) the closure of A generates a C_0-semigroup $(T(t))_{t\ge 0}$ which can be represented as limit of iterates of L_n, i.e. $T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f)$ for every $f \in E$ and every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$. Moreover, we have the following result.

Theorem 1.1.2 *Under the above assumptions, assume that D is a subspace of $\mathcal{D}$ such that for every $u \in D$ and $n \in \mathbb{N}$, we have*

$$\|n(L_n u - u)\| \le \varphi_n(u)\ , \tag{1.1.4}$$

and the following estimate of the Voronovskaja-type formula holds

$$\|n(L_n u - u) - Au\| \le \psi_n(u)\ , \tag{1.1.5}$$

where $\varphi_n, \psi_n : D \to [0, +\infty[$ are seminorms on the subspace D such that $\lim_{n\to\infty} \psi_n(u) = 0$ *for every $u \in D$.*

Then for every $t \ge 0$ and for every sequence $(k(n))_{n\ge1}$ of positive integers and $u \in D$, we have

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)} u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \\ &+ M \left(\exp(\omega\, e^{\omega/n}\, t_n) \left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n} \right. \\ &\left. + \frac{\omega}{n} \frac{k(n)}{n} \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n} \right) \right) \varphi_n(u)\end{aligned} \tag{1.1.6}$$

where $t_n := \sup\{t, k(n)/n\}$.

Proof. Let $n \ge 1$ and consider the linear bounded operator $A_n := n(L_n - I)$. It generates a uniformly continuous C_0-semigroup $(S_n(t))_{t\ge0}$ on E given by

$$S_n(t) = e^{t\,A_n} = e^{-nt}\, e^{nt\,L_n} = e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\,t)^k}{k!}\, L_n^k\ , \qquad t \ge 0\ .$$

Observe that $\mathcal{D}$ is a core for the closure of $(A, \mathcal{D})$ and consequently, from the first Trotter-Kato approximation theorem (see e.g. [48, Theorem 4.8, p. 209] and [64, Theorem III.4.4, p. 87]), we have that $(S_n(t))_{t\ge0}$ converges strongly to the C_0-semigroup $(T(t))_{t\ge0}$.

Consequently

$$\begin{aligned}\|S_n(t)\| &\le e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\,t)^k}{k!} \|L_n^k\| \le M e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\,t)^k}{k!}\, e^{\omega k/n} \\ &= M\, \exp\left(nt \left(e^{\omega/n} - 1 \right) \right)\ , \qquad t \ge 0\ .\end{aligned}$$

Now, let $(k(n))_{n\ge1}$ be an increasing sequence of positive integers and let $u \in D$. We have

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)} u\right\| &\le \|T(t)u - S_n(t)u\| + \left\| S_n(t)u - S_n\left(\frac{k(n)}{n}\right) u \right\| \\ &+ \left\| S_n\left(\frac{k(n)}{n}\right) u - L_n^{k(n)} u \right\|\ ,\end{aligned} \tag{1.1.7}$$

and therefore we can get (1.1.6) by estimating each term in (1.1.7).

As regards the first term we observe that, for every $n, m \geq 1$, we have

$$
\begin{aligned}
\|S_n(t)u - S_m(t)u\| &= \left\| \int_0^1 \frac{d}{ds} \left(e^{snt(L_n - I)}\, e^{(1-s)mt(L_m - I)} u \right) ds \right\| \\
&\leq \int_0^1 \left\| (nt(L_n - I) - mt(L_m - I))\, e^{snt(L_n - I)}\, e^{(1-s)mt(L_m - I)} u \right\| ds \\
&\leq t\, \|n(L_n - I)u - m(L_m - I)u\| \int_0^1 \|S_n(st)\|\, \|S_m((1-s)t)\|\, ds\ .
\end{aligned}
$$

If $\omega > 0$ we have

$$
\int_0^1 \|S_n(st)\|\, \|S_m((1-s)t)\|\, ds \leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{mt\left(e^{\omega/m}-1\right)}}{nt\left(e^{\omega/n}-1\right) - mt\left(e^{\omega/m}-1\right)}
$$

and hence

$$
\begin{aligned}
&\|S_n(t)u - S_m(t)u\| \\
&\quad \leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{mt\left(e^{\omega/m}-1\right)}}{nt\left(e^{\omega/n}-1\right) - mt\left(e^{\omega/m}-1\right)}\, t\, \|n(L_n - I)u - m(L_m - I)u\|\ .
\end{aligned}
$$

Taking the limit as $m \to +\infty$ and using the inequality

$$
e^x - 1 \leq x\, e^x\ , \qquad x \geq 0\ , \tag{1.1.8}
$$

we get

$$
\begin{aligned}
\|S_n(t)u - T(t)u\| &\leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \|n(L_n - I)u - Au\| \\
&\leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\
&\leq M^2\, \frac{e^{\omega t}\left(e^{nt\left(e^{\omega/n}-1\right)-\omega t} - 1\right)}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\
&\leq M^2\, \frac{e^{\omega t}\, t\left(n\left(e^{\omega/n}-1\right) - \omega\right) e^{nt\left(e^{\omega/n}-1\right)-\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\
&= M^2\, t e^{nt\left(e^{\omega/n}-1\right)}\, \psi_n(u) \\
&\leq M^2\, t e^{\omega t e^{\omega/n}}\, \psi_n(u)\ .
\end{aligned}
$$

Similarly, if $\omega = 0$ we obtain $\|S_n(t)u - T(t)u\| \leq M^2\, t\, \psi_n(u)$.

In regard to the second term in (1.1.7), using [64, Theorem I.2.4, d), p. 5] and (1.1.8), we have

$$
\begin{aligned}
\left\| S_n(t)u - S_n\left(\frac{k(n)}{n}\right) u \right\| &= \left\| \int_{k(n)/n}^{t} S_n(s)\,(n(L_n - I))u\,ds \right\| \\
&\le M\,\exp\left(n\,t_n\left(e^{\omega/n}-1\right)\right) \left|\frac{k(n)}{n} - t\right| \, \|n(L_n - I)u\| \\
&\le M\,e^{\omega\,t_n\,e^{\omega/n}} \left|\frac{k(n)}{n} - t\right| \varphi_n(u)\ ,
\end{aligned}
$$

where $t_n := \max\{t, k(n)/n\}$.

Finally, from Lemma 1.1.1 with $N = e^{\omega/n}$, $k = k(n)$ and $L = L_n$, we obtain

$$
\begin{aligned}
&\left\| S_n\left(\frac{k(n)}{n}\right) u - L_n^{k(n)} u \right\| = \left\| e^{k(n)(L_n - I)}u - L_n^{k(n)}u \right\| \\
&\le M\left(e^{\omega(k(n)-1)/n}\sqrt{\frac{2k(n)}{\pi}} + \frac{e^{k(n)(e^{\omega/n}-1)} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1} \right) \frac{\|n(Lu-u)\|}{n}\ .
\end{aligned}
$$

Using (1.1.8), we have

$$
\begin{aligned}
\frac{e^{k(n)(e^{\omega/n}-1)} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1} &\le \frac{e^{k(n)\,(\omega/n)\,e^{\omega/n}} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1} \\
&= \frac{e^{\omega\,k(n)/n}\left(e^{\omega\,(e^{\omega/n}-1)\,k(n)/n} - 1\right)}{e^{\omega/n}-1} \\
&\le \omega\,(e^{\omega/n}-1)\,\frac{k(n)}{n}\,\frac{e^{\omega\,k(n)/n}\,e^{\omega\,(e^{\omega/n}-1)k(n)/n}}{e^{\omega/n}-1} \\
&= \omega\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n}
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\left\| S_n\left(\frac{k(n)}{n}\right) u - L_n^{k(n)} u \right\| \\
&\qquad \le M\left(e^{\omega\,k(n)/n}\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n} + \frac{\omega}{n}\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n} \right) \varphi_n(u)\ .
\end{aligned}
$$

If $\omega = 0$, from (1.1.2) we get $\|S_n(k(n)/n)u - L_n^{k(n)}u\| \le M\,\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\,\varphi_n(u)$.
Hence, collecting the above inequalities, from (1.1.7) we get

$$
\begin{aligned}
\left\| T(t)u - L_n^{k(n)}u \right\| &\le M^2\,t\,e^{\omega\,t\,e^{\omega/n}}\,\psi_n(u) + M\,e^{\omega\,t_n\,e^{\omega/n}} \left|\frac{k(n)}{n} - t\right| \varphi_n(u) \\
&\quad + M\left(e^{\omega\,k(n)/n}\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n} + \frac{\omega}{n}\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n} \right) \varphi_n(u)
\end{aligned}
$$

if $\omega > 0$ and

$$\left\|T(t)u - L_n^{k(n)}u\right\| \le M^2\, t\, \psi_n(u) + M\, \left|\frac{k(n)}{n} - t\right|\, \varphi_n(u) + M\,\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\,\varphi_n(u)$$

if $\omega = 0$, as required. □

Remarks 1.1.3

1. From the preceding result, the order of convergence of $(L_n^{k(n)})_{n\ge1}$ to $T(t)$ on the subspace D depends on the order of convergence of $(\psi_n)_{n\ge1}$ (and eventually of $(\varphi_n)_{n\ge1}$) to 0, on the order of convergence of $|t - k(n)/n|_{n\ge1}$ to 0 (the best choice is $k(n) = [nt]$ which gives an order of convergence of $1/n$) and on $\sqrt{k(n)}/n$ which behaves like $\sqrt{t/n}$ as $n \to +\infty$. In most applications, an asymptotic behavior like $\sqrt{t/n}$ as $n \to +\infty$ can be obtained.

2. We can always take $\varphi_n(u) := \|Au\| + \psi_n(u)$ but we have preferred to introduce an independent estimate of $\|n(L_n - I)u\|$ since some applications can be more precise, as in the case of Stancu-Schnabl operators considered in the next section.

3. If the operators L_n are linear contractions, then the stability condition (1.1.3) on its iterates is automatically satisfied and from the representation (II.1.2) we obtain that the semigroup $(T(t))_{t\ge0}$ is itself of linear contractions.

4. If $u \in D$ and $L_n u = u$ for every $n \ge 1$, the proof of Theorem 1.1.2 also gives $T(t)u = u$ for every $t \ge 0$. Hence, the semigroup preserves every function which is preserved by all approximating operators.

5. Observe that estimate (1.1.6) holds uniformly with respect to t in compact intervals. □

In concrete applications, we often take $k(n) = [nt]$. In this case we obviously have $t_n = t$ and

$$\left|\frac{[nt]}{n} - t\right| = \frac{nt - [nt]}{n} \le \frac{1}{n}\,.$$

Hence estimate (1.1.6) yields

$$\begin{aligned}\left\|T(t)u - L_n^{[nt]}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \qquad (1.1.9)\\ &+ \frac{M}{\sqrt{n}}\left(\frac{\exp(\omega\, e^{\omega/n}\, t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\, e^{\omega t} + \frac{\omega\, t}{\sqrt{n}}\, \exp\left(\omega\, e^{\omega/n}\, t\right)\right)\varphi_n(u)\,.\end{aligned}$$

In most applications the growth bound ω of the semigroup will be equal to 0; in this particular case (1.1.6) and (1.1.9) become respectively

$$\left\| T(t)u - L_n^{k(n)} u \right\| \le M^2\, t\, \psi_n(u) + M \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{k(n)}}{n} \right) \varphi_n(u) \tag{1.1.10}$$

and

$$\left\| T(t)u - L_n^{[nt]} u \right\| \le M^2\, t\, \psi_n(u) + \frac{M}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \varphi_n(u)\,. \tag{1.1.11}$$

1.2 Estimate of the resolvent operator

1.2.1 Quantitative estimate of the convergence to the resolvent operator

The next result is concerned with the approximation of the resolvent operator of the closure of $(A, \mathcal{D})$.

If E is a real Banach space we consider the complexification $E_c \sim E \times E$ defined as usual by defining $(\alpha + i\beta)u := (\alpha u, \beta u)$ for every $\alpha, \beta \in \mathbb{R}$ and $u \in E$. An operator $L : E \to E$ can be regarded as acting on E_c by setting $L(u, v) = (L(u), L(v))$ for every $u, v \in E$.

At this point, for every $n \geq 1$ we define the linear operator $M_{\lambda,n} : E_c \to E_c$ as follows

$$M_{\lambda,n} u := \int_0^{+\infty} e^{-\lambda t} L_n^{[n\,t]} u \, dt \,, \qquad u \in E_c \,.$$

Theorem 1.2.1 *Consider the same assumptions of Theorem 1.1.2. If $\omega \geq 0$, then for every $n \geq 1$, $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$ and $u \in D$, we have*

$$\begin{aligned} \|R(\lambda, A)u - M_{\lambda,n} u\| \leq \frac{M^2}{(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\, \psi_n(u) + \frac{M}{\sqrt{n}} \Bigg(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})} \\ + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda - \omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2} \Bigg) \varphi_n(u) \,. \qquad (1.2.1) \end{aligned}$$

In particular, we have that the sequence $(M_{\lambda,n})_{n \geq 1}$ strongly converges to $R(\lambda, A)$.

Proof. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$. Using the integral representation of the resolvent operator (see, e.g., [48, Theorem II.1.10, (i), p. 55] and Chapter II) and taking into account (1.1.9) and the elementary properties of the gamma function, for every $n \geq 1$ and $u \in D$ we have

$$\|R(\lambda, A)u - M_{\lambda,n} u\| \leq \int_0^{+\infty} e^{-\operatorname{Re}\lambda\, t} \|T(t)u - L_n^{[nt]} u\| \, dt, \qquad (1.2.2)$$

then

$$\begin{aligned}\|R(\lambda,A)u - M_{\lambda,n}u\| &\le M^2\,\psi_n(u)\int_0^{+\infty} t\,\exp\Big((\omega e^{\omega/n}-\operatorname{Re}\lambda)\,t\Big)\,dt\\
&\quad+\frac{M}{n}\,\varphi_n(u)\int_0^{+\infty}\exp\Big((\omega e^{\omega/n}-\operatorname{Re}\lambda)\,t\Big)\,dt\\
&\quad+M\,\varphi_n(u)\sqrt{\frac{2}{\pi\,n}}\int_0^{+\infty}\sqrt{t}\,\exp\left((\omega-\operatorname{Re}\lambda)t\right)\,dt\\
&\quad+M\,\varphi_n(u)\,\frac{\omega}{n}\int_0^{+\infty} t\,\exp\Big((\omega e^{\omega/n}-\operatorname{Re}\lambda)\,t\Big)\,dt\\
&=\frac{M^2}{(\operatorname{Re}\lambda-\omega\,e^{\omega/n})^2}\,\psi_n(u)+\frac{M}{n(\operatorname{Re}\lambda-\omega\,e^{\omega/n})}\,\varphi_n(u)\\
&\quad+\frac{M}{\sqrt{2n}\,(\operatorname{Re}\lambda-\omega)^{3/2}}\,\varphi_n(u)+\frac{M\omega}{n(\operatorname{Re}\lambda-\omega\,e^{\omega/n})^2}\,\varphi_n(u)\ .\end{aligned}$$

Finally, the last part is a consequence of the density of $\mathcal{D}$ and the fact that estimate (1.2.2) imply that the sequence $(M_{\lambda,n}u)_{n\ge1}$ converges to $R(\lambda,A)u$ for every $u\in\mathcal{D}$. □

Remark 1.2.2 We explicitly point out that estimate (1.2.1) holds whenever $\operatorname{Re}\lambda>\omega$ if n is large enough (namely $n>\omega/\log(\operatorname{Re}\lambda/\omega)$), since $\lim_{n\to+\infty}e^{\omega/n}=1$. □

In the particular case $\omega=0$ we get, for every $n\ge1$, $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda>0$ and $u\in D$,

$$\|R(\lambda,A)u - M_{\lambda,n}u\| \le \frac{M^2}{(\operatorname{Re}\lambda)^2}\,\psi_n(u)+\frac{M}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\varphi_n(u)\ . \tag{1.2.3}$$

1.2.2 Approximation processes for resolvent operators

In this section we introduce some general sequences of linear operators, obtained from classical approximation processes, which approximate the resolvent operators of the generator of the C_0-semigroups.

The main aim is the possibility of representing the resolvent operators in terms of classical approximation operators.

First, we consider a sequence of $(L_n)_{n\ge1}$ of linear operators on a complex Banach space E (we consider its complexification if E is real) and assume that the hypotheses of Trotter's Theorem ore satisfied.

Now, let $(a_n)_{n\ge1}$ be a sequence of positive integers tending to $+\infty$. For every $n\ge1$, we consider the linear operator $P_{\lambda,a_n,n}:E\to E$ defined by

$$P_{\lambda,a_n,n}u := \frac{1}{n}\sum_{k=0}^{a_n} e^{-\lambda k/n}L_n^k u\ ,\qquad u\in E\ . \tag{1.2.4}$$

Theorem 1.2.3 *If the sequence $(a_n)_{n\geq 1}$ satisfies*

$$\lim_{n\to+\infty} \frac{a_n}{n} = +\infty \ , \tag{1.2.5}$$

then $\lim_{n\to+\infty} P_{\lambda,a_n,n} u = R(\lambda, A)u$ *for every* $u \in E$.

PROOF. Since $\operatorname{Re}\lambda > \omega$ we have $\left\| e^{-\lambda/n} L_n \right\| \leq M e^{-(\operatorname{Re}\lambda-\omega)/n}$ and consequently $\|P_{\lambda,a_n,n}\| \leq M/(1-e^{-(\operatorname{Re}\lambda-\omega)})$. Hence the sequence $(P_{\lambda,a_n,n})_{n\geq 1}$ is equibounded and we can show the convergence property on the dense subspace $\mathcal{D}$. Let $u \in \mathcal{D}$; from [1, (1.3)], we have

$$R(\lambda, A)u = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\infty} e^{-\lambda k/n} \frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u$$

and consequently

$$\begin{aligned} \|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \leq & \frac{1}{n} \left\| \sum_{k=0}^{a_n} e^{-\lambda k/n} L_n^k u - \sum_{k=0}^{\infty} e^{-\lambda k/n} \frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u \right\| \\ & + \left\| \frac{1}{n} \sum_{k=0}^{\infty} e^{-\lambda k/n} \frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u - R(\lambda,A)u \right\| \ . \end{aligned} \tag{1.2.6}$$

The second term in (1.2.6) tends to 0 by [1, (1.3)]. As regards to the first term, it is majored by

$$\begin{aligned} & \frac{M}{n} \|u\| \sum_{k=a_n+1}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n} + \frac{1}{n} \left\| \sum_{k=0}^{+\infty} e^{-\lambda k/n} \left(1 - \frac{1-e^{-\lambda/n}}{\lambda/n} \right) L_n^k u \right\| \\ & \leq \frac{M}{n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)} \|u\| \left(e^{-(\operatorname{Re}\lambda-\omega)(a_n+1)/n} + \left| 1 - \frac{1-e^{-\lambda/n}}{\lambda/n} \right| \right) \ . \end{aligned} \tag{1.2.7}$$

Since $\lim_{n\to+\infty} n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right) = \operatorname{Re}\lambda - \omega$, the assumption (1.2.5) ensures that the first term in (1.2.6) tends to 0 and this completes the proof. □

Our next aim is to provide a quantitative estimate in the above Theorem 1.2.3. Now we assume that there exist the seminorms $\varphi_n, \psi_n : D \to [0, +\infty[$ on a subspace D of $\mathcal{D}$ such that $\lim_{n\to\infty} \psi_n(u) = 0$ for every $u \in D$ and

$$\|n(L_n u - u)\| \leq \varphi_n(u) \ , \quad \|n(L_n u - u) - Au\| \leq \psi_n(u) \ . \tag{1.2.8}$$

Theorem 1.2.4 *Under assumptions (1.2.5) and (1.2.8), for every $n > \omega/\log(\operatorname{Re}\lambda/\omega)$ (or $n \geq 1$ if $\omega = 0$) and $u \in D$, we have*

$$\begin{aligned}\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \leq{}& \frac{M^2}{(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\,\psi_n(u) + \frac{M}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})}\right.\\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda-\omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\right)\varphi_n(u)\\ &+\frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{(\operatorname{Re}\lambda-\omega)\left(1 - \frac{\operatorname{Re}\lambda-\omega}{n}\right)}\,\|u\|\,. \end{aligned} \tag{1.2.9}$$

Proof. We estimate the two terms at the righthand side of (1.2.6). Set $\lambda = |\lambda|\, e^{i\theta}$; from the series expansion of $e^{-\lambda/n}$, we get

$$1 - \frac{1-e^{-\lambda/n}}{\lambda/n} = \sum_{k=2}^{+\infty}\frac{(-1)^k\,\lambda^{k-1}}{n^{k-1}k!} = -\sum_{k=1}^{+\infty}\frac{|\lambda|^k e^{ik(\theta+\pi)}}{n^k(k+1)!}\,. \tag{1.2.10}$$

Set for simplicity for every $k \in \mathbb{N}$, $b_k := \sum_{j=0}^{k} e^{ij(\theta+\pi)}$ and $c_k := |\lambda|^k/(n^k\cdot(k+1)!)$. For every $r \in \mathbb{N}$ we obtain

$$\sum_{k=1}^{r} c_k e^{ik(\theta+\pi)} = \sum_{k=1}^{r} c_k\, b_k - \sum_{k=1}^{r} c_k\, b_{k-1} = \sum_{k=1}^{r}(c_k - c_{k-1})\, b_k + c_{r+1}\, b_r - c_1\, b_0$$

and, letting $r \to +\infty$, $\left|\sum_{k=1}^{+\infty} c_k e^{ik(\theta+\pi)}\right| \leq \sum_{k=1}^{+\infty}(c_k - c_{k-1})\,|b_k| + c_1\,|b_0|$. Since

$$\begin{aligned}|b_k| = \left|\frac{1-e^{i(k+1)(\theta+\pi)}}{1-e^{i(\theta+\pi)}}\right| &\leq \frac{2}{\sqrt{(1+\cos\theta)^2+\sin^2\theta}} = \frac{2}{\sqrt{2(1+\cos\theta)}}\\ &= \frac{1}{|\cos(\theta/2)|}\end{aligned}$$

we conclude

$$\left|\sum_{k=1}^{+\infty} c_k e^{ik(\theta+\pi)}\right| \leq \frac{1}{|\cos(\theta/2)|}\left(\sum_{k=1}^{+\infty}(c_k - c_{k-1}) + c_1\right) = \frac{2c_1}{|\cos(\theta/2)|}\,.$$

Hence, from (1.2.10) we obtain

$$\left|1 - \frac{1-e^{-\lambda/n}}{\lambda/n}\right| \leq \frac{|\lambda|}{n\,|\sin((\theta+\pi)/2)|} = \frac{|\lambda|}{n\,|\cos(\theta/2)|} = \frac{|\lambda|^{3/2}}{n\,\left|\operatorname{Re}\sqrt{\lambda}\right|}\,. \tag{1.2.11}$$

Since $1 - e^x \geq -x(1+x)$ whenever $x \leq 0$, we have

$$\frac{1}{n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)} \leq \frac{1}{(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)} \tag{1.2.12}$$

and consequently, from (1.2.7) we get the following estimate of the first term in (1.2.6)

$$\frac{1}{n}\left\|\sum_{k=0}^{a_n} e^{-\lambda k/n} L_n^k u - \sum_{k=0}^{\infty} e^{-\lambda k/n}\,\frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u\right\| \le \frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n|\operatorname{Re}\sqrt{\lambda}|}\right)}{(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)}\|u\| \,.$$

In order to estimate the second term in (1.2.6), we consider the operator $M_{\lambda,n}$ introduced in the previous section and observe that

$$M_{\lambda,n}u := \int_0^{+\infty} e^{-\lambda\, t} L_n^{[n\,t]} u\, dt = \sum_{k=0}^{\infty} e^{-\lambda k/n}\,\frac{1-e^{-\lambda/n}}{\lambda}\, L_n^k u\,, \qquad u\in E$$

the last equality is valid since $L_n^{[nt]}$ is independent of t on each interval $[k/n,(k+1)/n[$. Then we can estimate the second term in (1.2.6) by Theorem 1.2.1

$$\|R(\lambda,A)u - M_{\lambda,n}u\| \le \frac{M^2}{(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\,\psi_n(u) + \frac{M}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda-\omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\right)\varphi_n(u)\,.$$

Using the above inequalities the proof is complete. □

Taking $a_n \ge [n\log n/\operatorname{Re}\lambda]$, estimate (1.2.9) becomes

$$\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \le C_1(\lambda)\,\psi_n(u) + \frac{C_2(\lambda)}{\sqrt{n}}\,\varphi_n(u) + \frac{C_3(\lambda)}{n}\,\|u\|\,, \quad (1.2.13)$$

for every $\omega\ge 0$, $u\in D$ and $n>\omega/\log(\operatorname{Re}\lambda/\omega)$ (or $n\ge 1$ if $\omega=0$), where $C_i(\lambda)$, $i=1,2,3$, are suitable constants depending only on λ.